Hale School

Mathematics Specialist

Test 1 --- Term 1 2016

Complex Numbers

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Name: **ANSWERS**

Instructions:

- **CAS calculators are NOT allowed**
- **External notes are not allowed**
- **Duration of test: 45 minutes**
- **Show your working clearly**
- **Use the method specified (if any) in the question to show your working (Otherwise, no marks awarded)**
- **This test contributes to 7% of the year (school) mark**

All arguments must be given using principal values.

Question 1 (4 marks: 1, 1, 1, 1)

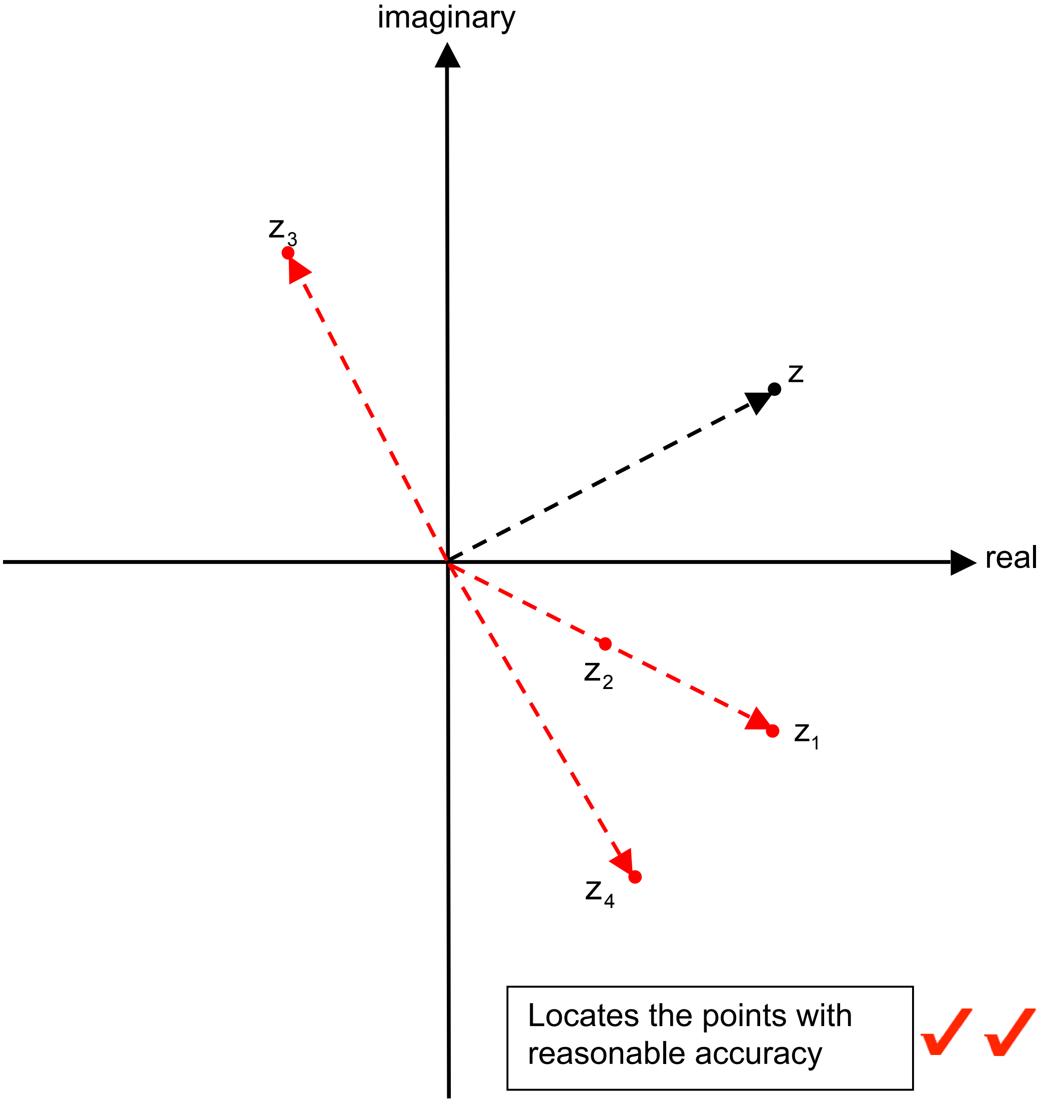
The following diagram shows a complex number z on the complex plane. Locate the following complex numbers. Label your answers clearly.

(a) $z_1 = \bar{z}$ (1 mark)

(b) $z_2 = \frac{1}{z}$ given that $|z|^2 = 2$ (1 mark)

(c) $z_3 = iz$ (1 mark)

(d) $z_4 = \frac{z}{i}$ (1 mark)



Locates the points with reasonable accuracy ✓✓✓✓

Question 2 (9 marks: 2, 3, 4)

(a) Convert $z = -1 + i$ to polar form. (2 marks)

$$|z| = \sqrt{1^2 + 1^1} = \sqrt{2}$$

$$\arg(z) = \tan^{-1}\left(\frac{1}{-1}\right) = \frac{3\pi}{4}$$

$$\text{So } z = \sqrt{2}\,\text{cis}\,\frac{3\pi}{4}$$

Determines the modulus	✓
Determines the argument	✓

(b) Determine the value(s) of θ and x if $z = 6 \text{ cis } \theta = x - 3i$. (3 marks)

$$z = 6\,\text{cis}\,\theta = x - 3i$$

$$\Rightarrow \quad 6\sin\theta = -3$$

$$\Rightarrow \quad \theta = -\frac{\pi}{6} \quad \text{or} \quad -\frac{5\pi}{6}$$

$$\text{So} \quad x = 3\sqrt{3} \quad \text{or} \quad -3\sqrt{3} \quad \text{respectively}$$

Sets up the correct equation	✓
Solves for θ correctly	✓
Solves for x	✓

(c) Determine the argument of $z = \dfrac{1}{\sin\frac{7\pi}{10} - i\cos\frac{7\pi}{10}}$. (4 marks)

$$z = \frac{1}{\sin\frac{7\pi}{10} - i\cos\frac{7\pi}{10}} \times \frac{\sin\frac{7\pi}{10} + i\cos\frac{7\pi}{10}}{\sin\frac{7\pi}{10} + i\cos\frac{7\pi}{10}}$$

$$= \sin\frac{7\pi}{10} + i\cos\frac{7\pi}{10}$$

$$= \sin\left(\frac{\pi}{2} + \frac{\pi}{5}\right) + i\cos\left(\frac{\pi}{2} + \frac{\pi}{5}\right)$$

$$= \cos\left(\frac{\pi}{5}\right) - i\sin\left(\frac{\pi}{5}\right)$$

$$= \cos\left(-\frac{\pi}{5}\right) + i\sin\left(-\frac{\pi}{5}\right)$$

$$\text{So} \quad \arg(z) = -\frac{\pi}{5}$$

Realises z	✓
Changes sin to cos etc	✓
Simplifies correctly	✓
States the answer	✓

Question 3 (8 marks: 4, 4)

(a) **Given** that if $z = \cos\theta + i\sin\theta$ then $z^n = \cos(n\theta) + i\sin(n\theta)$ is true for all positive integers.

Show that $z^n = \cos(n\theta) + i\sin(n\theta)$ is also true for all **negative** integers. (4 marks)

Let $m = -n$ be a negative integer

$$z^m = z^{-n} = (z^n)^{-1}$$
$$= [\cos(n\theta) + i\sin(n\theta)]^{-1}$$
$$= \frac{1}{\cos(n\theta) + i\sin(n\theta)} \times \frac{\cos(n\theta) - i\sin(n\theta)}{\cos(n\theta) - i\sin(n\theta)}$$
$$= \cos(n\theta) - i\sin(n\theta)$$
$$= \cos(-n\theta) + i\sin(-n\theta)$$
$$= \cos(m\theta) + i\sin(m\theta)$$

So $z^n = \cos(n\theta) + i\sin(n\theta)$ is true for $n \in Z$

Writes $z^m = (z^n)^{-1}$	✓
Uses the given result	✓
Realises the complex number	✓
Simplifies correctly	✓

(b) Use De Moivre's theorem to show that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$. (4 marks)

Consider $z^3 = (\cos\theta + i\sin\theta)^3$

Using De Moivre's theorem:

$$z^3 = \cos 3\theta + i\sin 3\theta$$

Using binomial theorem:

$$z^3 = \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3$$
$$= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)$$

Equate i part:

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$$
$$\Rightarrow \quad \sin 3\theta = 3(1 - \sin^2\theta)\sin\theta - \sin^3\theta$$
$$\therefore \quad \sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

Expands z^3 using two different methods	✓
Simplifies and collects terms in the binomial expansion	✓
Equates the imaginary parts	✓
Simplifies correctly	✓

Question 4 (7 marks: 2, 2, 3)

It is given that **one** of the roots of the complex equation $z^3 = u$ is $2 \text{ cis } \frac{2\pi}{5}$.

(a) Write down all the other roots in polar form. (2 marks)

$$z_2 = 2\,\text{cis}\,(-\frac{14\pi}{15})$$

$$z_3 = 2\,\text{cis}\,(-\frac{4\pi}{15})$$

Shows angle $2\pi/3$ between roots ✓
Shows same modulus for each ✓

(b) Determine the complex number u. (2 marks)

$$u = (2\,\text{cis}\,\frac{2\pi}{5})^3$$

$$= 8\,\text{cis}\,(-\frac{4\pi}{5})$$

Raises any root to power 3 ✓
Simplifies correctly ✓

(c) Use the given information that $2 \text{ cis } \frac{2\pi}{5}$ is a solution of $z^3 = u$ to determine **one** solution of the complex equation $z^3 = -ui$. (3 marks)

$$z^3 = -u\,i$$

$$\Rightarrow \quad \frac{z^3}{-i} = u$$

$$\Rightarrow \quad (\frac{z}{i})^3 = u$$

$$\Rightarrow \quad \frac{z}{i} = 2\,\text{cis}\,(\frac{2\pi}{5})$$

$$\Rightarrow \quad z = 2\,\text{cis}\,(\frac{9\pi}{10})$$

Other roots: $2\,\text{cis}\,(\frac{7\pi}{30})$ and $2\,\text{cis}\,(-\frac{13\pi}{30})$

Rearranges the equation to the form $Z^3 = u$ ✓
Uses the given solution correctly ✓
States the answer ✓

Question 5 (5 marks: 2, 3)

(a) The locus of z such that | z – (1 + i) | = | z – (2 – 4i) | can be interpreted geometrically as the set of points equidistant from (1, 1) and (2, -4).

Give a similar geometrical interpretation of | z – (1 + i) | = | 2z – (2 – 4i) |. (2 marks)

$$|z-(1+i)| = |2z-(2-4i)|$$
$$\Rightarrow \quad |z-(1+i)| = 2|z-(1-2i)|$$

So z is the set of points such that its distance from (1, 1) is twice that from (1, – 2)

Writes 2z – (2 – 4i) as 2[z – (1 – 2i)]	✓
Provides the correct interpretation	✓

(b) Determine the Cartesian equation of | z – (1 + i) | = | 2z – (2 – 4i) |. (3 marks)

$$|z-(1+i)| = |2z-(2-4i)|$$
$$\Rightarrow \quad (x-1)^2+(y-1)^2 = 4[(x-1)^2+(y+2)^2]$$
$$\Rightarrow \quad 3x^2-6x+3y^2+18y+18=0$$
$$\Rightarrow \quad 3(x-1)^2+3(y+3)^2=12$$
$$\Rightarrow \quad (x-1)^2+(y+3)^2=4$$

Circle centred at (1, – 3) with radius = 2.

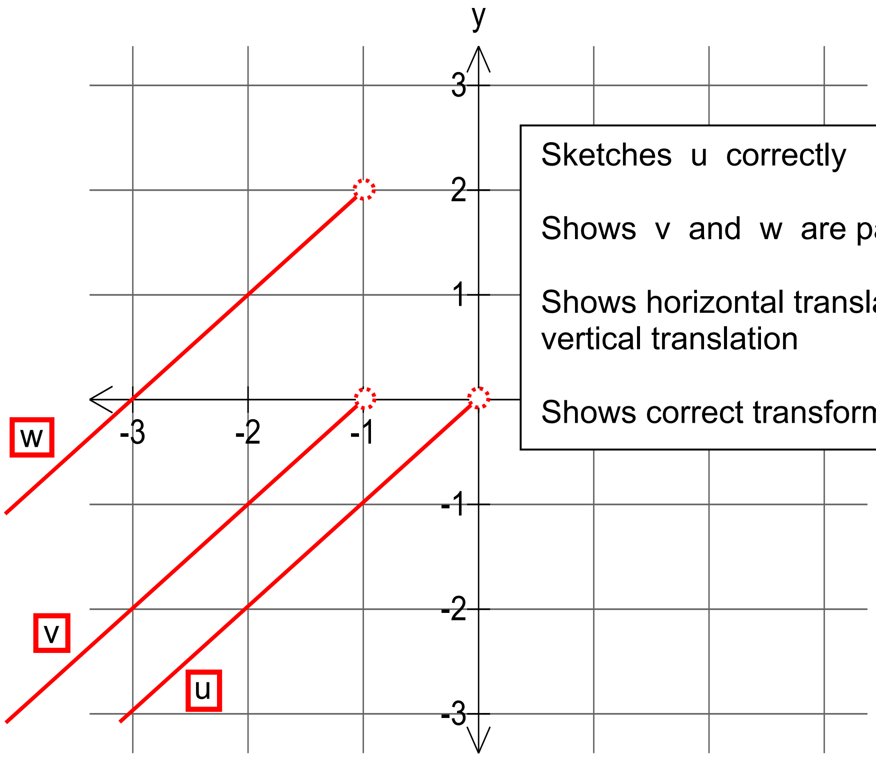
Expresses the modulus of 2z – (2 – 4i) correctly	✓
Sets up the equation	✓
Simplifies correctly	✓

Question 6 (6 marks: 4, 2)

(a) Sketch the locus of u, v and w if

(i) $\arg(u) = -\frac{3\pi}{4}$ (ii) $\arg(v+1) = -\frac{3\pi}{4}$ (iii) $\arg(w+1-2i) = -\frac{3\pi}{4}$

Label the loci clearly. (4 marks)



Sketches u correctly ✓
Shows v and w are parallel to u ✓
Shows horizontal translation and vertical translation ✓
Shows correct transformations ✓

(b) Determine z such that $\arg(z+1-2i) = -\frac{3\pi}{4}$ and $|z|$ is a minimum. (2 marks)

$\arg(z+1-2i)$ and $|z|$ is a min

$\Rightarrow$ (vector) $z \perp$ (locus) w

$\Rightarrow$ $z = -\frac{3}{2} + \frac{3}{2}i$

Recognizes that $z \perp w$ ✓
States the answer ✓

Question 7 (4 marks: 2, 2)

(a) Show that $(z - i)$ is a factor of $P(z) = z^3 + iz^2 + (2 - 7i)z - 7$. (2 marks)

$P(z) = z^3 + iz^2 + (2 - 7i)z - 7$
$P(i) = (i)^3 + i(i)^2 + (2 - 7i)(i) - 7$
$= -2i + 2i + 7 - 7$
$= 0$

$\therefore$ $(z - i)$ is a factor of $P(z)$.

Substitutes $z = i$ into $P(z)$ ✓
Shows $P(i) = 0$ ✓

(b) Form a polynomial with **integer** coefficients such that $x = \sqrt{2}$ and $x = i$ are two of the roots of the polynomial. (2 marks)
(Note: The polynomial has other roots.)

$x = \sqrt{2}$ and $x = i$
$\Rightarrow$ $x^2 = 2$ and $x^2 = -1$
$\Rightarrow$ $(x^2 - 2)$ and $(x^2 + 1)$ are factors
$\Rightarrow$ $P(x) = (x^2 - 2)(x^2 + 1)$
$\therefore$ $P(x) = x^4 - x^2 - 2$

Realises $x = -i$ is also a root and forms $(x - i)(x + i)$ ✓
Shows the other factor $(x^2 - 2)$ ✓